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INTERACTION OF SHOCK WAVES DUE TO COMBINED TWO SHEAR LOADINGS

T. C. T. TING*

University of Illinois at Chicago Circle, Chicago, Illinois 60680

Abstract—The propagation of plane elastic–plastic waves due to combined two shear loadings is studied in which the materials are assumed to be elastic, isotropic work hardening. For a general strain hardening law, analytical solutions can be obtained only for simple waves and for two shear waves which are reducible to a single shear wave. If the material is elastic, linearly strain hardening, analytical solutions can also be obtained for two shear waves due to a series of step loadings and unloadings. The solution consists of shock waves propagating at constant speeds with a constant stress state between the shocks. The interaction of two or more shocks meeting at a point, and the reflection of a shock wave from a rigid surface or from an interface between two different media are studied in detail. The results can be applied to two shear waves in a plate of finite thickness which consists of two or more layers of different materials. It is shown briefly that similar results are obtained for materials with elastic, kinematical work hardening behavior.

1. INTRODUCTION

THERE has been increasing interest in the study of elastic–plastic wave propagation of combined stresses in recent years (see, for example, [1–6]). A typical problem is that of wave propagation of combined normal and shear stresses considered by Bleich and Nelson [3], and Clifton [4], and the two shear waves studied by Fong [5]. Assuming isotropic work-hardening, the wave propagation is governed by a system of quasilinear hyperbolic partial differential equations of first order with two pairs of characteristics whose slopes are finite and non-zero. For combined normal and shear stresses, these two characteristic slopes are not constants in the plastic regions even if the material is assumed to be elastic, linearly strain hardening in a simple shear test. The characteristic condition along a characteristic line cannot be integrated and Rieman invariants, which exist for hyperbolic systems with one pair of characteristics, do not exist here. Therefore, except for particular cases such as simple wave solutions [4], analytical solutions are very difficult to obtain and the problem can be solved only approximately by a numerical approach [6].

If the elastic–plastic wave propagation is induced by a loading of two shears which are perpendicular to each other and uniformly distributed on one surface of an infinite plate, the governing system of hyperbolic differential equations yields two characteristic slopes whose values correspond to the elastic shear wave speed and the plastic shear wave speed in a simple shear test. Since the elastic shear wave speed is a constant, at least one pair of the characteristics has a constant slope. If the material behaves in an elastic, linearly strain hardening manner in a simple shear test, the plastic wave speed is also a constant and both pairs of characteristics are straight lines with constant slopes. This is why the two shear wave problem provides an attractive example for the analysis of wave propagation for combined stresses.

* Associate Professor of Applied Mechanics, Department of Materials Engineering.

For a general strain hardening material in which the stress–strain curve for a simple shear test is concave to the strain axis, analytical solutions can be obtained for the following initial and boundary value problems:

(i) The ratio of the two time-dependent shears applied at one surface of the plate is a constant. If the initial values are not zero, the ratios of the two shears and the two velocities prescribed initially must be the same constant as the one prescribed on the boundary.

(ii) Both initial and boundary values are constants. The ratio of the two shears at the boundary, and the ratios of the two shears and the two velocities initially need not necessarily be the same.

The first case is clearly reducible to a single shear wave problem and hence the theory developed by Karman, Taylor and Rakhmatulin applies. The second case can be solved by the simple wave solutions. In both cases, analytical solutions are possible only for a plate of infinite thickness, i.e. for a half-space, if the solution is to be valid for all time. (Additional restriction may be required on the boundary values of the first case if a complete analytical solution is needed.) Also in both cases, it can be shown that the ratio of the two shears during a plastic loading is a constant.

When the ratio of the two shears is a constant during a plastic loading, the characteristic conditions become integrable. If the material is elastic, linearly strain hardening, an analytical solution can be obtained for two shear wave propagation in a plate of finite thickness due to a series of step loadings and unloadings. The plate may consist of two or more layers of different materials. The solution consists of shock waves propagating at the elastic or plastic wave speed with constant stress state between the shocks. Thus, the interaction of two or more shocks at a point, and reflection of a shock from a rigid surface, from a free surface, and from an interface between two different media becomes the essence of the solution.

In the following, the system of differential equations for the combined two shear wave propagation is derived in Section 2 and the corresponding characteristics and the characteristic conditions are obtained. In Section 3, the simple wave solution for the combined two shear wave propagation is given. A particular case of the simple wave solution in which the material is elastic, linearly strain hardening is presented in Section 4. It is shown that the simple wave solution for elastic, linearly strain hardening material consists of three regions of constant stress state with shock waves as their boundaries. In Section 5, the interaction of two or more shock waves meeting at a point and the reflection of a shock wave from an interface between two different media are given. This is then applied in two examples in Section 6 for combined two shear waves in an elastic, linearly strain hardening material due to a series of step loadings and unloadings. The solutions can be obtained analytically or graphically. Finally, in Section 7, an analysis is given briefly for materials which exhibit kinematical work-hardening. It is shown that the essential features of the solution for isotropic work-hardening materials remain the same for kinematical work-hardening.

2. THE BASIC EQUATIONS AND THE CHARACTERISTIC CONDITIONS

Consider an infinite plate which occupies $0 \leq x \leq h$ of a Cartesian coordinate x, y, z where h is a constant. If h is infinite, we have a half-space. Of three displacements u_x, u_y, u_z we consider the motion in which $u_x \equiv 0$ while u_y and u_z are functions of x and time t only. Thus, the only nonvanishing stresses and strains are $\tau_{xy}, \tau_{xz}, \varepsilon_{xy}$, and ε_{xz}. For simplicity, we will use the following notations:

$$\begin{aligned} \tau_1 &= \tau_{xy} & \tau_2 &= \tau_{xz}, \\ \gamma_1 &= 2\,\varepsilon_{xy} & \gamma_2 &= 2\,\varepsilon_{xz}, \\ v_1 &= \dot{u}_y & v_2 &= \dot{u}_z, \end{aligned} \tag{1}$$

where a dot stands for $\partial/\partial t$. The equations of motion are

$$\frac{\partial \tau_1}{\partial x} = \rho \dot{v}_1, \tag{2a}$$

$$\frac{\partial \tau_2}{\partial x} = \rho \dot{v}_2, \tag{2b}$$

where ρ is the mass density of the plate. For an elastic, isotropic work-hardening material, the stress–strain law is (see [7]).

$$\begin{aligned} \dot{\varepsilon}_{ij} &= \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij} \\ &= \frac{1}{2\mu}\dot{\tau}_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\delta_{ij}\dot{\tau}_{kk} + G(k)\frac{\partial f}{\partial \tau_{ij}}\frac{\partial f}{\partial \tau_{km}}\dot{\tau}_{km}. \end{aligned} \tag{3}$$

Here λ and μ are the Lamé constants, $f(\tau_{ij})$ is the yield condition, and k is the yield stress. For the two shear problem considered here, both the von Mises and Tresca yield conditions give

$$f = \tau_1^2 + \tau_2^2 = k^2, \tag{4}$$

and equation (3) yields

$$\frac{1}{2}\dot{\gamma}_1 = \frac{1}{2\mu}\dot{\tau}_1 + G(k)\tau_1(2\tau_1\dot{\tau}_1 + 2\tau_2\dot{\tau}_2), \tag{5a}$$

$$\frac{1}{2}\dot{\gamma}_2 = \frac{1}{2\mu}\dot{\tau}_2 + G(k)\tau_2(2\tau_1\dot{\tau}_1 + 2\tau_2\dot{\tau}_2). \tag{5b}$$

$G(k)$ in equation (5) can be obtained from the stress–strain relation of a simple shear test. In simple shear, the relation between τ and γ is, by letting $\tau_2 = 0$ and omitting the subscript 1 in equation (5a),

$$\frac{1}{2}\dot{\gamma} = \frac{1}{2\mu}\dot{\tau} + G(k)2\tau^2\dot{\tau}.$$

But $\dot{\tau} = (\mathrm{d}\tau/\mathrm{d}\gamma)\dot{\gamma}$ and $\tau^2 = k^2$ in simple shear. Hence

$$G(k) = \frac{1}{4k^2}\left(\frac{1}{\mu_p} - \frac{1}{\mu}\right), \tag{6}$$

where $\mu_p = \mathrm{d}\tau/\mathrm{d}\gamma$ is the slope of the $\tau \sim \gamma$ relation in the plastic region. For a given $\tau \sim \gamma$ relation we can obtain μ_p in terms of τ. Since $\tau = k$ for simple shear, μ_p in (6) is $\mu_p(k)$. With the continuity conditions

$$\dot{\gamma}_1 = \frac{\partial v_1}{\partial x}, \qquad \dot{\gamma}_2 = \frac{\partial v_2}{\partial x}, \tag{7}$$

Equations (2) and (5) can be rewritten as

$$\mathbf{A}\dot{\mathbf{w}}+\mathbf{B}\frac{\partial \mathbf{w}}{\partial x} = 0, \tag{8}$$

where

$$\mathbf{A} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{1}{\mu}+\frac{1}{k^2}\tau_1^2\left(\frac{1}{\mu_p}-\frac{1}{\mu}\right) & 0 & \frac{1}{k^2}\tau_1\tau_2\left(\frac{1}{\mu_p}-\frac{1}{\mu}\right) \\ 0 & 0 & \rho & 0 \\ 0 & \frac{1}{k^2}\tau_1\tau_2\left(\frac{1}{\mu_p}-\frac{1}{\mu}\right) & 0 & \frac{1}{\mu}+\frac{1}{k^2}\tau_2^2\left(\frac{1}{\mu_p}-\frac{1}{\mu}\right) \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} v_1 \\ \tau_1 \\ v_2 \\ \tau_2 \end{bmatrix}.$$

Equation (8) is a system of quasi-linear first order partial differential equations. Notice that both matrices **A** and **B** are symmetric.

The characteristic slope c of equation (8) is obtained by (see [8])

$$|c\mathbf{A}-\mathbf{B}| = 0. \tag{9}$$

The four roots of equation (9), denoted by $\pm c_e$ and $\pm c_p$, are

$$c_e^2 = \frac{\mu}{\rho}, \qquad c_p^2 = \frac{\mu_p}{\rho}, \tag{10}$$

c_e corresponds to the elastic wave speed while c_p corresponds to the plastic wave speed. The characteristic condition along a characteristic is obtained by

$$\mathbf{l}^T\mathbf{A}(\mathrm{d}\mathbf{w}) = 0, \tag{11}$$

where $\mathbf{l}^T$ is the transpose of the left eigenvector $\mathbf{l}$ of the equations

$$\mathbf{l}^T(c\mathbf{A}-\mathbf{B}) = 0, \tag{12}$$

and $\mathrm{d}\mathbf{w}$ is the total differential of $\mathbf{w}$. Omitting lengthy but otherwise straight forward calculations, we write the characteristic conditions equation (11) in the following form:

$$\tau_2(\mathrm{d}\tau_1 \mp \rho c_e\,\mathrm{d}v_1)-\tau_1(\mathrm{d}\tau_2 \mp \rho c_e\,\mathrm{d}v_2) = 0 \qquad \text{along } c = \pm c_e, \tag{13a}$$

$$\tau_1(\mathrm{d}\tau_1 \mp \rho c_p\,\mathrm{d}v_1)+\tau_2(\mathrm{d}\tau_2 \mp \rho c_p\,\mathrm{d}v_2) = 0 \qquad \text{along } c = \pm c_p. \tag{13b}$$

If we introduce vector notations

$$\boldsymbol{\tau} = (\tau_1, \tau_2), \qquad \mathbf{v} = (v_1, v_2), \tag{14}$$

Equations (13) can be written as

$$\boldsymbol{\tau} \times (\mathrm{d}\boldsymbol{\tau} \mp \rho c_e\,\mathrm{d}\mathbf{v}) = 0 \qquad \text{along } c = \pm c_e, \tag{15a}$$

$$\boldsymbol{\tau} \cdot (\mathrm{d}\boldsymbol{\tau} \mp \rho c_p\,\mathrm{d}\mathbf{v}) = 0 \qquad \text{along } c = \pm c_p. \tag{15b}$$

The analysis presented so far made no assumption concerning the strain hardening property μ_p except that μ_p be a non-increasing function of k. Thus the characteristic conditions as expressed by (13) or (15) hold for general strain hardening materials.

For unloading, the material behaves in an elastic manner. By letting $c_p = c_e$ in equations (13) or (15) we deduce the characteristic conditions for the elastic region:

$$\left.\begin{aligned} d\tau_1 \mp \rho c_e\, dv_1 &= 0 \\ d\tau_2 \mp \rho c_e\, dv_2 &= 0 \end{aligned}\right\} \text{along } c = \pm c_e. \tag{16}$$

Hence, if there is a discontinuity in stresses and velocities across the characteristics $c = c_e$ or $c = -c_e$, the discontinuities must satisfy the following jump conditions

$$\begin{aligned} [\tau_1] \pm \rho c_e [v_1] &= 0 \\ [\tau_2] \pm \rho c_e [v_2] &= 0 \end{aligned} \qquad \text{on } c = \pm c_e, \tag{17a}$$

or, in vector notation

$$[\boldsymbol{\tau}] \pm \rho c_e [\mathbf{v}] = 0 \qquad \text{on } c = \pm c_e, \tag{17b}$$

where $[f]$ stands for the discontinuity of f.

3. THE SIMPLE WAVE SOLUTION

Although the theory on which simple wave solutions are based has been established and applied in other fields [9], it does not seem to have been used widely in wave propagation of combined stresses. We will present briefly the theory of simple wave solutions of equation (8) and derive the generalized Rieman's invariants in this section. In the problem considered here, there is only one simple wave solution, but it will be clear from the analysis presented below that the results can be easily extended to more general systems of hyperbolic partial differential equations.

We define a simple wave solution by the particular solution of equation (8) in which the dependent variable $\mathbf{w} = (v_1, \tau_1, v_2, \tau_2)$ is a constant vector along a characteristic line. Since the characteristic slope c is a function of τ_1 and τ_2, this implies that the characteristics are straight lines for simple wave solutions. Now, $\mathbf{w}$ = constant along a line with slope $dx/dt = c$ implies

$$\frac{\partial \mathbf{w}}{\partial x} c + \dot{\mathbf{w}} = 0. \tag{18}$$

By eliminating $\dot{\mathbf{w}}$, between equations (18) and (8), we obtain

$$(c\mathbf{A} - \mathbf{B}) \frac{\partial \mathbf{w}}{\partial x} = 0. \tag{19a}$$

On the other hand, elimination of $\partial \mathbf{w}/\partial x$ gives

$$(c\mathbf{A} - \mathbf{B})\dot{\mathbf{w}} = 0 \tag{19b}$$

Since $d\mathbf{w} = a(\partial \mathbf{w}/\partial x) + b\dot{\mathbf{w}}$ is the total differential along any direction $dx/dt = a/b$, if a and b are arbitrary, we can combine equations (19a) and (19b) and write

$$(c\mathbf{A} - \mathbf{B})\, d\mathbf{w} = 0. \tag{20}$$

Thus, if $\mathbf{r}$ is the right eigenvector of the equation

$$(c\mathbf{A}-\mathbf{B})\mathbf{r} = 0, \tag{21}$$

$\mathbf{dw}$ is proportional to $\mathbf{r}$. If $(c\mathbf{A}-\mathbf{B})$ is a symmetric matrix as is the case in the present problem, the left eigenvector $\mathbf{l}$ and the right eigenvector $\mathbf{r}$ are identical. Hence $\mathbf{dw}$ is proportional to $\mathbf{l}$. For $c = +c_p$, $\mathbf{l}$ of equation (12) can be shown to be

$$\mathbf{l} = \begin{vmatrix} \tau_1 \\ -\rho c_p \tau_1 \\ \tau_2 \\ -\rho c_p \tau_2 \end{vmatrix} \quad \text{for} \quad c = c_p, \tag{22}$$

and the fact that $\mathbf{dw}$ is proportional to $\mathbf{l}$ gives

$$\frac{\mathrm{d}v_1}{\tau_1} = \frac{\mathrm{d}\tau_1}{-\rho c_p \tau_1} = \frac{\mathrm{d}v_2}{\tau_2} = \frac{\mathrm{d}\tau_2}{-\rho c_p \tau_2}. \tag{23}$$

Equation (23) is equivalent to three ordinary differential equations. Integration of these equations gives

$$\tau_1 = \alpha\tau_2, \tag{24a}$$

$$v_1 = \alpha v_2 + \beta, \tag{24b}$$

$$v_2 = -\int^{\tau_2} \frac{\mathrm{d}\tau_2}{\rho c_p(k)} + \delta, \tag{24c}$$

where α, β, δ are constants. Since $k^2 = \tau_1^2 + \tau_2^2 = (1+\alpha^2)\tau_2^2$, equation (24c) can be written as

$$v_2 = -\frac{1}{\sqrt{(1+\alpha^2)}} \int^{\sqrt{(1+\alpha^2\tau_2)}} \frac{\mathrm{d}k}{\rho c_p(k)} + \delta.$$

Equations (24) are the generalized Rieman's invariants which hold in the region where the simple wave solution applies.

Now suppose that in equation (8) the initial and boundary conditions are given by

$$\mathbf{w}(x,0) = \begin{bmatrix} v_{1i} \\ \tau_{1i} \\ v_{2i} \\ \tau_{2i} \end{bmatrix} \quad 0 \le x \le \infty, \tag{25a}$$

$$\boldsymbol{\tau}(0,t) = \begin{bmatrix} \tau_{1b} \\ \tau_{2b} \end{bmatrix} \quad 0 < t, \tag{25b}$$

where $v_{1i}, \tau_{1i}, v_{2i}, \tau_{2i}, \tau_{1b}$ and τ_{2b} are constants. Let

$$\tau_{1i}^2 + \tau_{2i}^2 = k_i^2, \qquad \tau_{1b}^2 + \tau_{2b}^2 = k_b^2, \tag{26}$$

and $k_b > k_i$ so that we have plastic loading. In the x–t plane (Fig. 1a), the solution can be divided into three regions. The region between the line OA and the x-axis is a constant

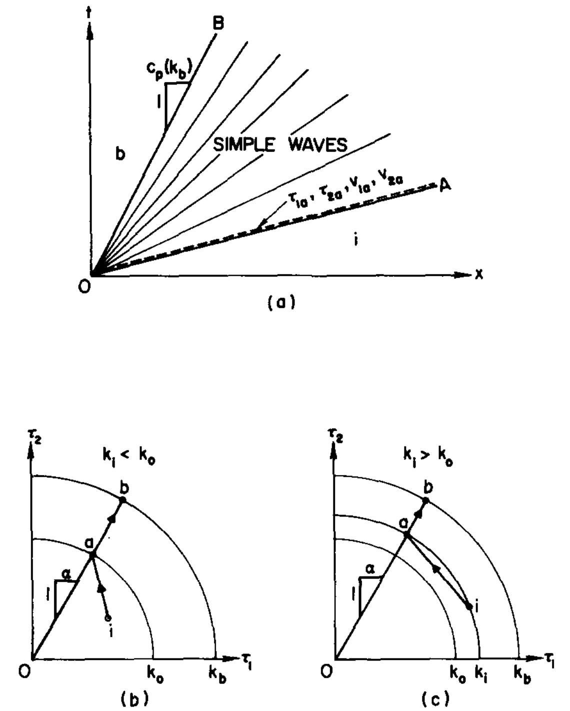

FIG. 1

state with stresses and velocities given by equation (25a). The region between the line OB and the t-axis is another constant state in which the stresses are given by equation (25b). The region between OA and OB is where the simple wave solution applies. While the solution is continuous across OB, it is in general discontinuous across OA. Let the subscript a denote the solution along the top of line OA. Then by equations (17a), we obtain

$$\begin{aligned}(\tau_{1a}-\tau_{1i})+\rho c_e(v_{1a}-v_{1i}) &= 0,\\ (\tau_{2a}-\tau_{2i})+\rho c_e(v_{2a}-v_{2i}) &= 0.\end{aligned} \tag{27}$$

To complete the formulation, we have

$$\tau_{1a}^2+\tau_{2a}^2 \begin{cases} = k_0^2 & \text{if} \quad k_i < k_0 \\ = k_i^2 & \text{if} \quad k_i \geq k_0 \end{cases} \tag{28}$$

where k_0 is the initial yield stress.

Equations (24) to (28) are sufficient to determine all the unknowns. In the τ_1–τ_2 plane (Figs. 1b and 1c), the stress point i jumps to point a at the elastic wave front OA (Fig. 1a) and moves continuously from point a to point b as the last plastic wave OB arrives. Notice that the stress path ab is a portion of the radial line ob with the slope α (equation (24a)).

Hence, with vector notations, we can write

$$\boldsymbol{\tau}_a = n\boldsymbol{\tau}_b, \tag{29}$$

where n is a constant.

4. SIMPLE WAVES IN LINEARLY STRAIN HARDENING MATERIALS

We will now assume that the material is elastic, linearly strain hardening so that c_p is a constant. For the initial and boundary values given by equations (25), the solution in the x–t plane consists of three regions of constant state (see Fig. 2a). The stresses and velocities

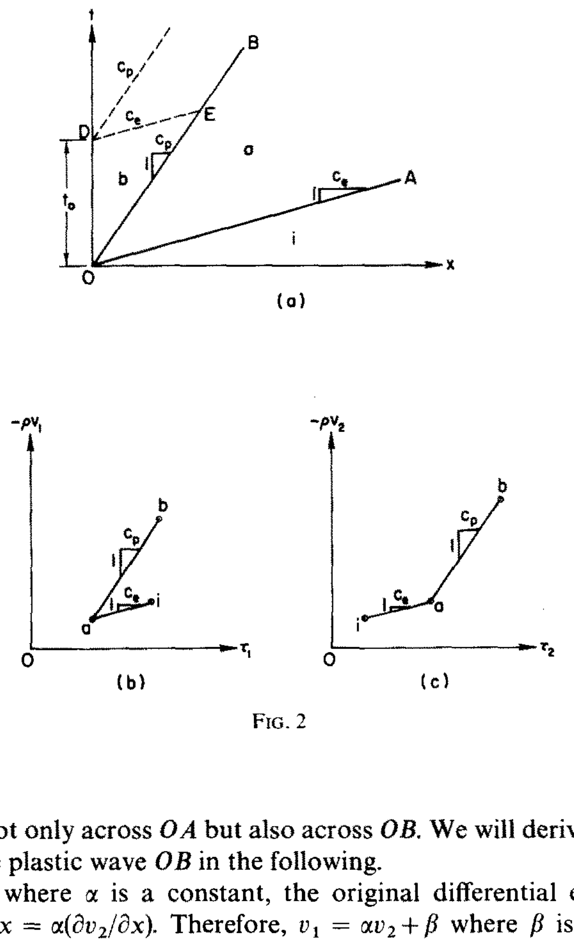

FIG. 2

are discontinuous not only across OA but also across OB. We will derive the discontinuity condition across the plastic wave OB in the following.

When $\tau_1 = \alpha\tau_2$ where α is a constant, the original differential equation (8) yields $\dot{v}_1 = \alpha\dot{v}_2$ and $\partial v_1/\partial x = \alpha(\partial v_2/\partial x)$. Therefore, $v_1 = \alpha v_2 + \beta$ where β is another constant. With $\tau_1 = \alpha\tau_2$ and $v_1 = \alpha v_2 + \beta$, the characteristic condition (13a) is automatically satisfied while (13b) is reduced to

$$\left.\begin{aligned} \mathrm{d}\tau_1 \mp \rho c_p \, \mathrm{d}v_1 &= 0 \\ \mathrm{d}\tau_2 \mp \rho c_p \, \mathrm{d}v_2 &= 0 \end{aligned}\right\} \quad \text{along} \quad c = \pm c_p.$$

Since c_p is now a constant, we have,

$$\begin{aligned} \tau_1 \mp \rho c_p v_1 &= \text{const.} \\ \tau_2 \mp \rho c_p v_2 &= \text{const.} \end{aligned} \qquad \text{along} \quad c = \pm c_p.$$

Therefore, if there is a discontinuity across $c = \pm c_p$, we must have

$$\begin{aligned} [\tau_1] \pm \rho c_p [v_1] &= 0 \\ [\tau_2] \pm \rho c_p [v_2] &= 0 \end{aligned} \qquad \text{on} \quad c = \pm c_p, \tag{30a}$$

or, in vector notations

$$[\boldsymbol{\tau}] \pm \rho c_p [\mathbf{v}] = 0 \qquad \text{on} \quad c = \pm c_p. \tag{30b}$$

This is the condition which has to be satisfied across the line OB in Fig. 2a.

With the initial and boundary values given by equations (25), we can obtain the stresses in the region a and velocities in the regions a and b graphically as follows. Since $\boldsymbol{\tau}_b$ is given, point b in the stress plane can be located (see Figs. 1b and 1c). By connecting point b with the origin, point a is obtained by the intersection of ob and the circle of radius k_0 or k_i depending on whether $k_i < k_0$ or $k_i > k_0$. Thus we obtain τ_{1_a} and τ_{2_a}. Now the jump condition across OA (Fig. 2a) as given by equations (17a) or equations (27) indicates that the line connecting points i and a in the $\tau_1 \sim (-\rho v_1)$ plane and the $\tau_2 \sim (-\rho v_2)$ plane has the slope of c_e (Fig. 2b and 2c). Similarly, the jump condition across OB as given by equations (30a) indicates that the line connecting points a and b in the $\tau_1 \sim (-\rho v_1)$ plane and the $\tau_2 \sim (-\rho v_2)$ plane has the slope of c_p. With the knowledge of stresses at a and b, the points a and b in Figs. 2(b) and 2(c) can be located graphically. The ordinates of a and b in these two figures then give the velocities at a and b.

If, in Fig. 2(a), the stresses prescribed at the boundary $x = 0$ are changed to a new constant value after $t = t_0$, a new simple wave solution can be obtained with point D (Fig. 2a) as the new origin. Thus we will have again two shock waves, one traveling at the speed c_e and the other traveling at the speed c_p with a constant state between them. Since $c_e > c_p$, the shock wave DE eventually intersects the shock wave OB at point E (Fig. 2a). In the next section, we will study the interaction of two or more shock waves meeting at one point.

5. INTERACTION OF SHOCK WAVES

In Figs. 3, suppose that there are two or more shock waves converging to the point Q as indicated by the dotted lines and, as a result, new shock waves are generated as shown by the solid lines. To be more general, we will assume that the material to the left of point Q is different from the material to the right of point Q. Therefore, PQ is the interface between the two materials. The particular case in which a shock wave reflects from a rigid boundary can be obtained by either letting $c_e' = \infty$, $c_p' = \infty$, or by letting $c_e = c_e'$, $c_p = c_p'$ and considering wave motions which are symmetric with respect to the line PQ.

The stresses and the velocities in the regions 1 and 1′ are known and we will determine the stresses and the velocities in the regions 2, 3, 2′ and 3′. We will use vector notations $\boldsymbol{\tau}$ and $\mathbf{v}$ for the stress and the velocity respectively and the subscripts 1, 1′, 2, ..., 3′ to denote the regions. Thus τ_1 is the stress in region 1 and v'_2 is the velocity in region 2'. Also, we will denote by k_m and k'_m the maximum yield stress ever reached by the material to the left and right respectively of point Q.

Depending on the strength of the incoming shock waves and the values of k_m, k'_m, the new shock waves generated will have one of the four different patterns as shown in Figs. 3a–d. We will discuss them separately in the following.

Case I $k_3 \geq k_m, k'_m$ (Fig. 3a). For this case, the discontinuity in stress and velocity between regions 1 and 2 is, by equation (17b),

$$\tau_1 - \rho c_e v_1 = \tau_2 - \rho c_e v_2, \tag{31}$$

and the discontinuity between regions 2 and 3 is, by equation (30b)

$$\tau_2 - \rho c_p v_2 = \tau_3 - \rho c_p v_3. \tag{32}$$

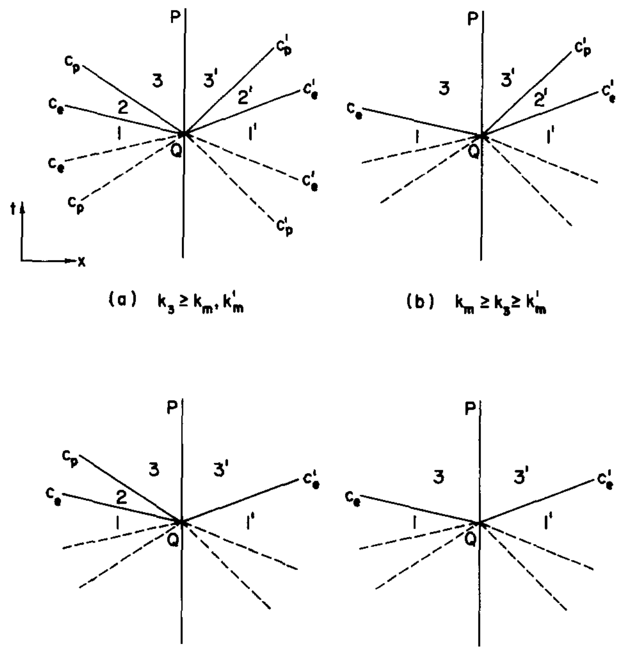

FIG. 3

Similarly, consideration of the jump conditions between regions 1' and 2' and between 2' and 3' gives

$$\tau_{1'} + \rho c'_e v_{1'} = \tau_{2'} + \rho c'_e v_{2'}, \tag{33}$$

$$\tau_{2'} + \rho c'_p v_{2'} = \tau_3 + \rho c'_p v_3, \tag{34}$$

where use has been made of the continuity relation $\tau_3 = \tau_{3'}$, $v_3 = v_{3'}$. Now, the yield stress k_2 in region 2 cannot be larger than k_m, the maximum yield stress, since from region 1 to region 2 the process is elastic. On the other hand, the process from region 2 to region 3 is plastic so that the stress state of region 2 must be at the yield surface k_m. Hence

$$k_2 = |\boldsymbol{\tau}_2| = k_m. \tag{35}$$

Similarly,

$$k_{2'} = |\boldsymbol{\tau}_{2'}| = k'_m. \tag{36}$$

Moreover, by equation (29), we also have

$$\boldsymbol{\tau}_2 = n\boldsymbol{\tau}_3, \tag{37}$$

$$\boldsymbol{\tau}_{2'} = n'\boldsymbol{\tau}_3, \tag{38}$$

where n and n' are constants.

Equations (31) to (38) give a complete description of the shock interaction. From equations (35) to (38) one obtains

$$\boldsymbol{\tau}_2 = \frac{k_m}{k_3}\boldsymbol{\tau}_3, \tag{39}$$

$$\boldsymbol{\tau}_{2'} = \frac{k'_m}{k_3}\boldsymbol{\tau}_3. \tag{40}$$

Elimination of $\mathbf{v}_2$, $\mathbf{v}_{2'}$ and $\mathbf{v}_3$ from equations (31) to (34) gives, with the use of equations (39) and (40),

$$\left(\frac{1}{c_e}+\frac{1}{c'_e}\right)\mathbf{K} = \left\{\left[\frac{1}{c_p}-\left(\frac{1}{c_p}-\frac{1}{c_e}\right)\frac{k_m}{k_3}\right]+\left[\frac{1}{c'_p}-\left(\frac{1}{c'_p}-\frac{1}{c'_e}\right)\frac{k'_m}{k_3}\right]\right\}\boldsymbol{\tau}_3, \tag{41}$$

where

$$\mathbf{K} = \frac{1}{\dfrac{1}{c_e}+\dfrac{1}{c'_e}}\left[\frac{1}{c_e}(\boldsymbol{\tau}_1-\rho c_e\mathbf{v}_1)+\frac{1}{c'_e}(\boldsymbol{\tau}_{1'}+\rho c'_e\mathbf{v}_{1'})\right]. \tag{42}$$

From equation (41) it is clear that $\boldsymbol{\tau}_3$ is proportional to $\mathbf{K}$. If we let $K = |\mathbf{K}|$, we have

$$\boldsymbol{\tau}_3 = \frac{k_3}{K}\mathbf{K}. \tag{43}$$

On the other hand, by taking the absolute values of both sides of equation (41), we obtain

$$k_3 = \frac{1}{\dfrac{1}{c_e}+\dfrac{1}{c'_e}}\left\{\left(\frac{1}{c_e}+\frac{1}{c'_e}\right)K+\left(\frac{1}{c_p}-\frac{1}{c_e}\right)k_m+\left(\frac{1}{c'_p}-\frac{1}{c'_e}\right)k'_m\right\}. \tag{44}$$

Therefore, equations (42), (44), (43), (40), (39), (31), (32), (33), in the order mentioned, provide the complete solution of the shock interaction.

The solution presented above is valid as long as k_3 as expressed by equation (44) is larger than k_m and k'_m.

Case II $k_m \geq k_3 \geq k'_m$ (Fig. 3b). In this case, there is no shock wave with the speed c_p. The jump condition between regions 1 and 3 is, by equation (17b),

$$\boldsymbol{\tau}_1-\rho c_e\mathbf{v}_1 = \boldsymbol{\tau}_3-\rho c_e\mathbf{v}_3. \tag{45}$$

Equations (45), (33), (34), (36) and (38) give a complete description of the shock interaction for this case. By a similar analysis as in Case I, it can be shown that equations (40) and (43) still hold and equation (44) is replaced by

$$k_3 = \frac{1}{\dfrac{1}{c_e}+\dfrac{1}{c'_p}}\left\{\left(\frac{1}{c_e}+\frac{1}{c'_e}\right)K+\left(\frac{1}{c'_p}-\frac{1}{c'_e}\right)k'_m\right\}, \tag{46}$$

where $K = |\mathbf{K}|$ is defined in equation (42).

Case III $k'_m \geq k_3 \geq k_m$ (Fig. 3c). In this case, the jump condition between regions 1′ and 3′ is, by equation (17b)

$$\boldsymbol{\tau}_{1'} + \rho c'_e \mathbf{v}_{1'} = \boldsymbol{\tau}_{3'} + \rho c'_e \mathbf{v}_3. \tag{47}$$

Equations (47), (31), (32), (35) and (37) give a complete description of the shock interaction. As in Case II, it can be shown that equations (39) and (43) still hold and equation (44) is replaced by

$$k_3 = \frac{1}{\dfrac{1}{c'_e}+\dfrac{1}{c_p}}\left\{\left(\frac{1}{c_e}+\frac{1}{c'_e}\right)K+\left(\frac{1}{c_p}-\frac{1}{c_e}\right)k_m\right\}, \tag{48}$$

where $K = |\mathbf{K}|$ is defined in equation (42).

Case IV $k_3 \leq k_m, k'_m$ (Fig. 3d). For this case, only two elastic shock waves exist. The jump condition between regions 1 and 3 is

$$\boldsymbol{\tau}_1 - \rho c_e \mathbf{v}_1 = \boldsymbol{\tau}_3 - \rho c_e \mathbf{v}_3,$$

while the jump condition between regions 1′ and 3′ is

$$\boldsymbol{\tau}_{1'} + \rho c'_e \mathbf{v}_{1'} = \boldsymbol{\tau}_3 + \rho c'_e \mathbf{v}_3.$$

Elimination of $\mathbf{v}_3$ yields

$$\boldsymbol{\tau}_3 = \mathbf{K}, \tag{49}$$

and hence

$$k_3 = |\mathbf{K}| = K, \tag{50}$$

where $\mathbf{K}$ is defined in equation (42).

It can be shown that the four cases discussed above are mutually exclusive, i.e. there is a unique solution for a given set of $\boldsymbol{\tau}_1$, $\mathbf{v}_1$, $\boldsymbol{\tau}_{1'}$, $\mathbf{v}_{1'}$ and k_m, k'_m. This is also evident from a graphical solution which we will present below.

The vector $\mathbf{K}$ defined by equation (42) and used by all four cases can be obtained graphically as follows. For a given $\boldsymbol{\tau}_1, \mathbf{v}_1, \boldsymbol{\tau}_{1'}, \mathbf{v}_{1'}$ we locate the points 1 and 1′ in the $\tau_1 \sim (-\rho v_1)$ plane and the $\tau_2 \sim (-\rho v_2)$ plane (see Figs. 4a and 4b). In both planes, draw a straight line with positive slope c'_e from point 1′ and a straight line with negative slope c_e from point 1. Denote the abscissae of the intersections of these two lines by K_1 and K_2 respectively. Then $\mathbf{K} = (K_1, K_2)$. (The proof is omitted here.) In Case IV, $\boldsymbol{\tau}_3 = \mathbf{K}$ so that $\boldsymbol{\tau}_3$ is determined. In the other three cases, $\boldsymbol{\tau}_3$ is proportional to $\mathbf{K}$ so that if we know k_3, we can determine $\boldsymbol{\tau}_3$ and hence the point 3 in the stress plane (see Fig. 4c).

To obtain k_3 for the remaining three cases, let us consider Case I first. k_3 as expressed by equation (44) can be obtained graphically in the following manner. On rectangular

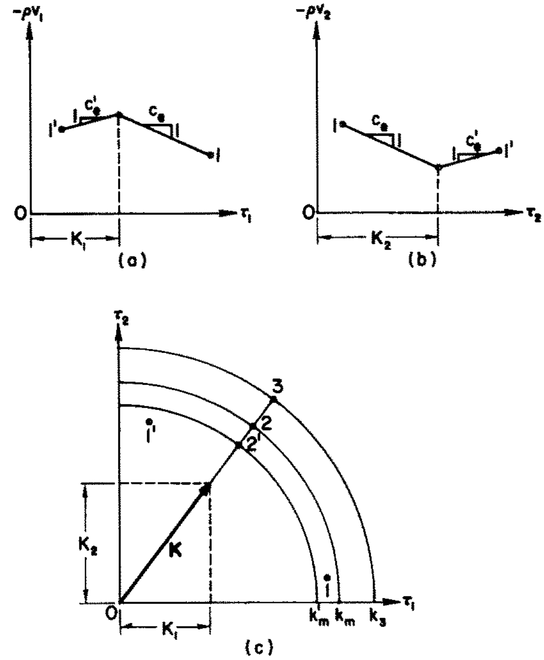

FIG. 4

coordinates, (Fig. 5a), we locate K, k_m and k'_m on the horizontal axis whose abscissae correspond to their magnitudes. Without loss of generality, we assume that $k_m > k'_m$.

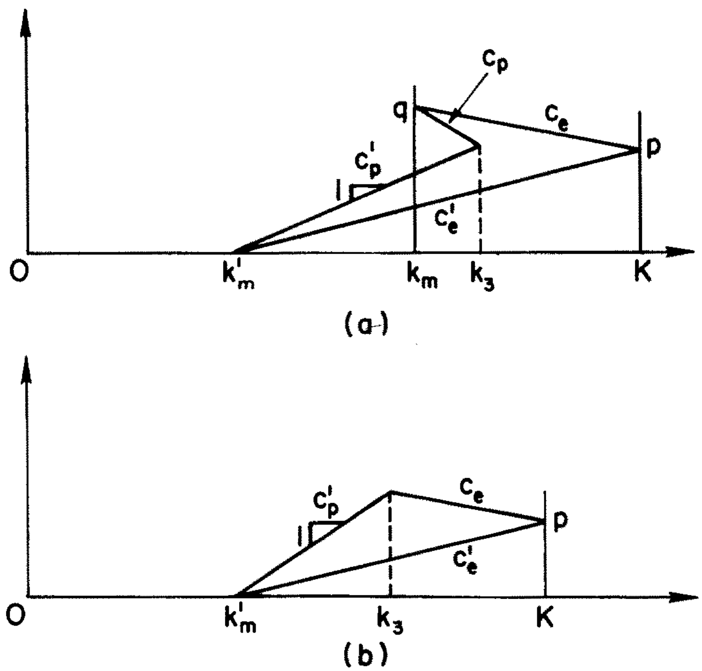

FIG. 5

Draw a straight line with positive slope c'_e from k'_m which intersects the vertical line through K at p. From p draw a straight line with negative slope c_e which intersects the vertical line through k_m at q. Finally, draw a straight line from q with negative slope c_p and a straight line from k'_m with positive slope c'_p. It can be shown that the abscissa of the intersection of these two lines is k_3 as expressed by equation (44).

For Case II, the procedure is the same as in Case I up to the point p (Fig. 5b). Then we draw a straight line from p with negative slope c_e and a straight line from k'_m with positive slope c'_p. It can be shown that the abscissa of the intersection of these two lines is k_3 as expressed by equation (46).

For Case III, the procedure of determining k_3 is exactly the same as that shown in Fig. 5(b) if we replace k'_m, c'_e, c'_p, c_e in Fig. 5(b) by k_m, c_e, c_p, c'_e respectively.

After determination of k_3 and $\boldsymbol{\tau}_3, \boldsymbol{\tau}_2$ and $\boldsymbol{\tau}_{2'}$ as expressed by equations (37) to (40) can be obtained graphically (cf. Figs. 1b and 1c). In Fig. 4(c), $\boldsymbol{\tau}_2$ and $\boldsymbol{\tau}_{2'}$ are determined for Case I. The procedure is self-explanatory. For other cases the procedures are similar. The velocities $\mathbf{v}_2, \mathbf{v}'_2$, and $\mathbf{v}_3$ as expressed by equations (31) to (34) can also be determined graphically (cf. Figs. 2b and 2c). Thus a complete graphical solution can be obtained.

The case in which a shock wave is reflected from a free surface is trivial and is omitted here.

6. EXAMPLES

Consider a half-space in which the material is elastic, linearly strain hardening. For illustrative purposes, we assume that the initial yield stress $k_0 = 1000$ psi and take the ratio $c_e/c_p = 4$. As a first example, consider the initial and boundary conditions as given by

$$\begin{aligned} &\tau_1(x,0) = v_1(x,0) = v_2(x,0) = 0, \\ &\tau_2(x,0) = 915\text{ psi}, \qquad 0 \le x < \infty, \\ &\tau_1(0,t) \begin{cases} = 2000\text{ psi}, & 0 < t < t_0, \\ = 0, & t_0 < t, \end{cases} \\ &\tau_2(0,t) = 915\text{ psi}, \qquad 0 < t. \end{aligned} \tag{51}$$

The complete solution is shown in Figs. 6. It is seen in Fig. 6(a) that the plastic shock wave OB does not penetrate into region 17. It is completely absorbed at point B. In Fig. 6(b), the stress state in each region is shown, while in Fig. 6(c) and 6(d), the velocities are given.

The shear strains in each region can be obtained easily by the following considerations. The continuity of displacements across a shock wave requires that

$$\begin{aligned} [\mathbf{v}] \pm c_e[\boldsymbol{\gamma}] = 0 \qquad \text{on} \quad c = \pm c_e, \\ [\mathbf{v}] \pm c_p[\boldsymbol{\gamma}] = 0 \qquad \text{on} \quad c = \pm c_p, \end{aligned} \tag{52}$$

where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$. Making use of equations (17b) and (30b), equation (52) can be written as

$$\begin{aligned} \rho c_e^2[\boldsymbol{\gamma}] = [\boldsymbol{\tau}] \qquad \text{on} \quad c = \pm c_e, \\ \rho c_p^2[\boldsymbol{\gamma}] = [\boldsymbol{\tau}] \qquad \text{on} \quad c = \pm c_p. \end{aligned} \tag{53}$$

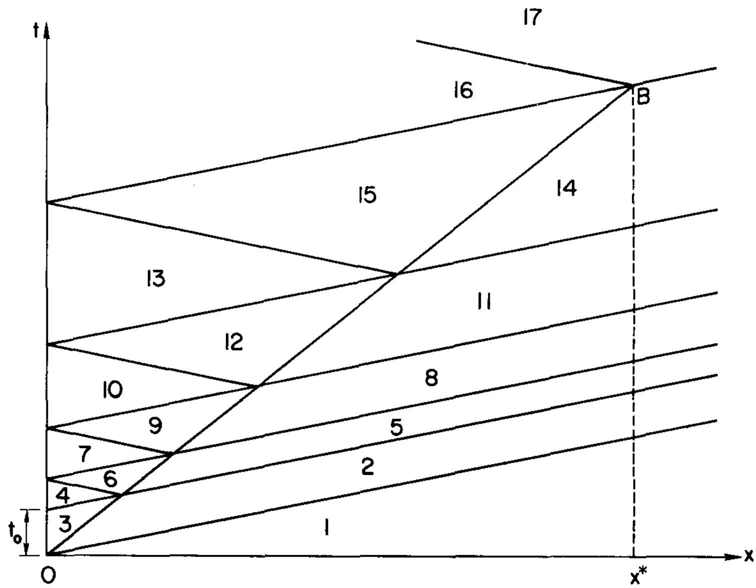
t
17
16
B
15
14
13
11
12
10
8
9
5
7
6
2
4
3
1
t_0
0
x*
x

FIG. 6a

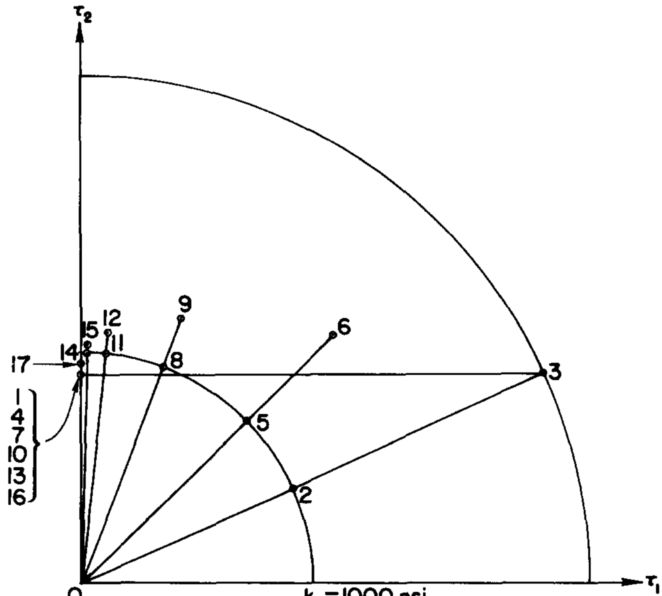
τ_2
9
12
6
15
11
14
17
8
3
1
4
7
10
13
16
5
2
0
k_0=1000 psi
τ_1

FIG. 6b

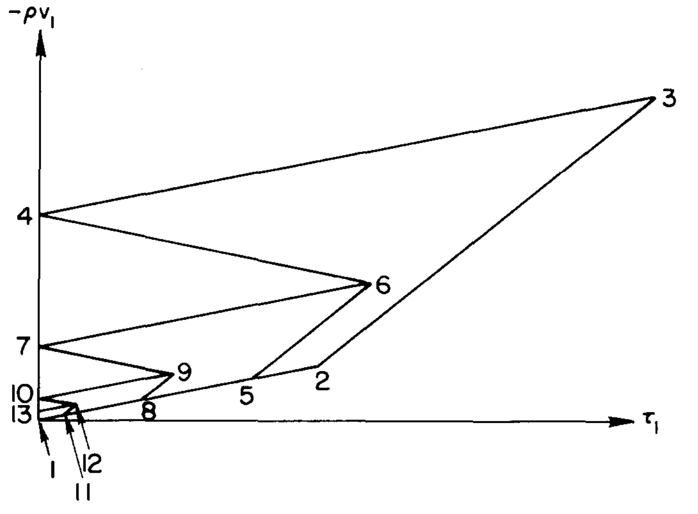

FIG. 6c

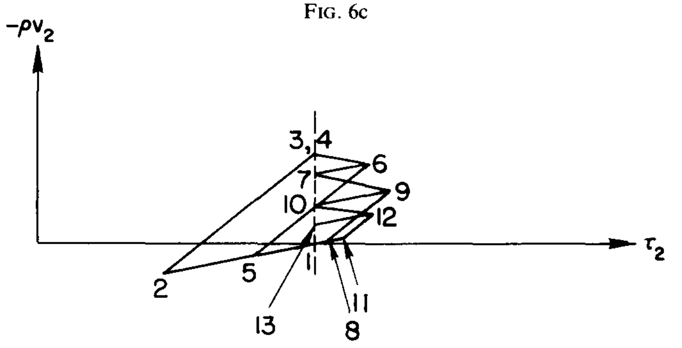

FIG. 6d

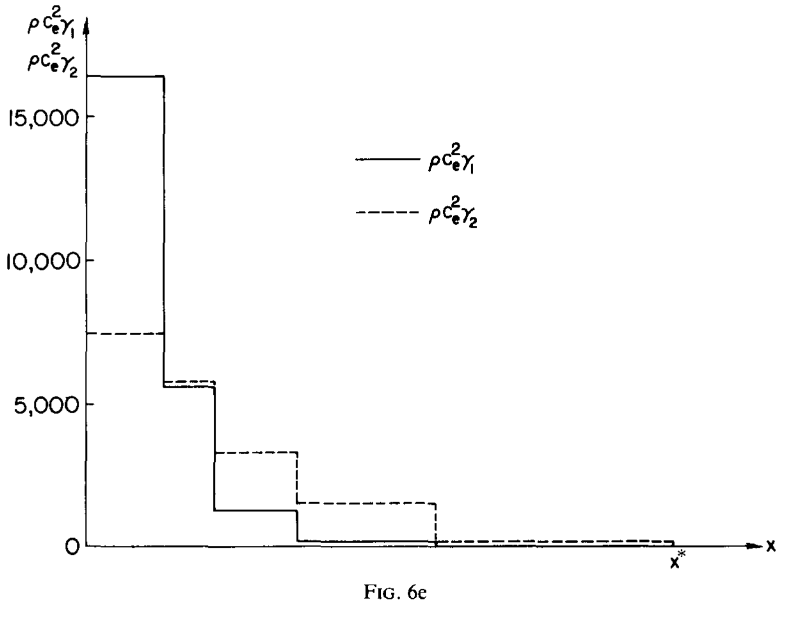

FIG. 6e

Thus shear strains in each region can be obtained by either equations (52) or (53). In Fig. 6(e), the residual plastic strains are given by using equation (53).

Let us consider the second example in which the initial and boundary values are given by

$$\begin{aligned} \tau_1(x,0) &= v_1(x,0) = v_2(x,0) = 0, \\ \tau_2(x,0) &= 1000\,\text{psi}, \qquad 0 \le x < \infty, \\ \tau_1(0,t) &\begin{cases} = 750\,\text{psi}, & 0 < t \le t_0, \\ = 800\,\text{psi}, & t_0 < t, \end{cases} \\ \tau_2(0,t) &= 1000\,\text{psi}, \qquad 0 < t. \end{aligned} \tag{54}$$

In other words, the half-space is prestressed to the yield limit by τ_2 and, at $t = 0$, an additional stress $\tau_1 = 750$ psi is added to the boundary. After a duration of time t_0, this additional stress τ_1 is increased to 800 psi. The shock wave interaction for this example is shown in Fig. 7(a). It is seen that even though the material is prestressed to the yield limit, the initial disturbance is propagated at the elastic wave speed c_e as indicated by the line OA. (A similar result was found by Clifton [4].) Moreover, at $t = t_0$ when the applied force is increased, the disturbance is still propagated at the elastic wave speed instead of the plastic wave speed as shown by the line DB. The dotted lines in Fig. 6(a) are also shock waves but their discontinuities are so insignificant that their existence can be ignored. In Fig. 7(b), the stresses are given for each region.

7. TWO SHEAR WAVES IN KINEMATICAL WORK-HARDENING MATERIALS

For kinematical work-hardening materials, we express the yield condition for the two shear problem in the following form

$$f = (\tau_1 - \tau_1^*)^2 + (\tau_2 - \tau_2^*)^2 = k^2 \tag{55}$$

where k is now a constant. With the flow rule

$$\dot{\gamma}_1^p = \dot{\Lambda}(\tau_1 - \tau_1^*), \qquad \dot{\gamma}_2^p = \dot{\Lambda}(\tau_2 - \tau_2^*),$$

we have the relation

$$\dot{\tau}_1^* = b\dot{\Lambda}(\tau_1 - \tau_1^*) \tag{56a}$$

$$\dot{\tau}_2^* = b\dot{\Lambda}(\tau_2 - \tau_2^*) \tag{56b}$$

where b is also a constant as suggested by Prager [10]. $\dot{\Lambda} > 0$ if the material undergoes plastic deformations and $\dot{\Lambda} = 0$ if the material behaves elastically. The stress–strain law now yields

$$\dot{\gamma}_1 = \frac{1}{\mu}\dot{\tau}_1 + \dot{\Lambda}(\tau_1 - \tau_1^*), \tag{57a}$$

$$\dot{\gamma}_2 = \frac{1}{\mu}\dot{\tau}_2 + \dot{\Lambda}(\tau_2 - \tau_2^*). \tag{57b}$$

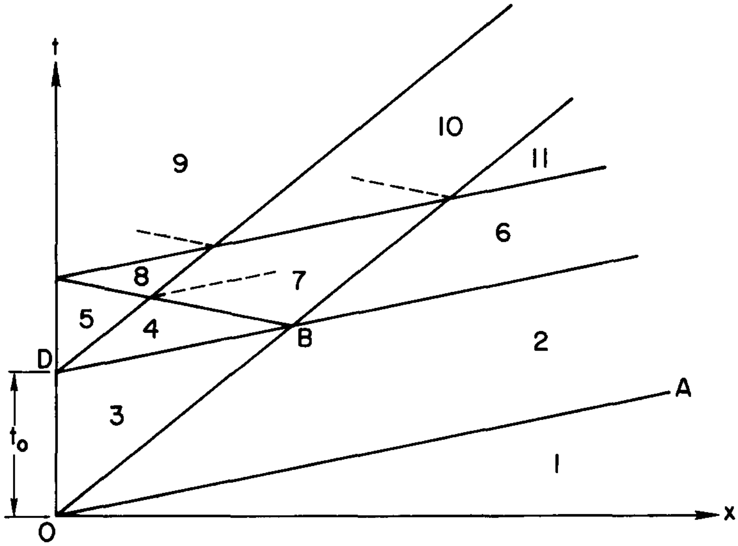

FIG. 7a

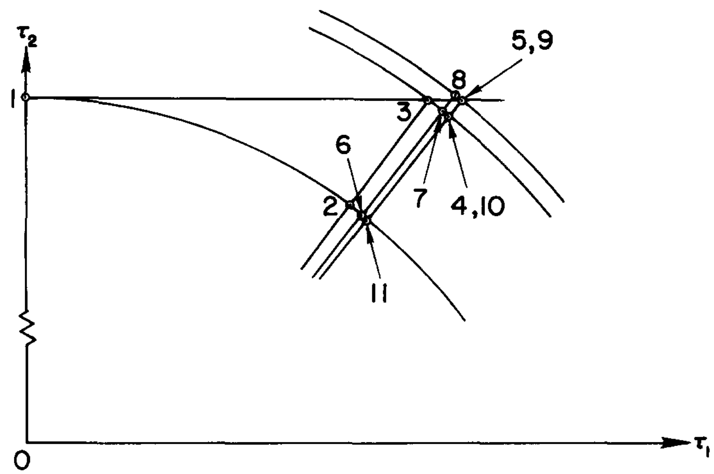

FIG. 7b

In simple shear, the constant b is related to μ_p, the work-hardening coefficient, by the equation

$$\frac{1}{b} = \frac{1}{\mu_p} - \frac{1}{\mu}. \tag{58}$$

The complete system of equations governing the two shear waves in kinematical work-hardening materials consists of equations (2) and (55) to (57) which can be written as, with the aids of equation (7),

$$\mathbf{A}\dot{\mathbf{w}} + \mathbf{B}\frac{\partial \mathbf{w}}{\partial x} = 0, \tag{59}$$

where

$$\mathbf{A} = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\mu} & 0 & 0 & 0 & 0 & \bar{\tau}_1 \\ 0 & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 & \bar{\tau}_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -b\bar{\tau}_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -b\bar{\tau}_2 \\ 0 & \bar{\tau}_1 & 0 & \bar{\tau}_2 & -\bar{\tau}_1 & -\bar{\tau}_2 & 0 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} v_1 \\ \tau_1 \\ v_2 \\ \tau_2 \\ \tau_1^* \\ \tau_2^* \\ \Lambda \end{bmatrix},$$

and

$$\bar{\tau}_1 = \tau_1 - \tau_1^*, \qquad \bar{\tau}_2 = \tau_2 - \tau_2^* \tag{60}$$

Following the analysis presented before, it can be shown that of seven characteristic slopes of equation (59), three have zero slope and the rest of four have the slopes expressed by equation (10). The characteristic conditions along $c = \pm c_e$ and $c = \pm c_p$ are

$$\bar{\tau}_2(\mathrm{d}\tau_1 \mp \rho c_e \,\mathrm{d}v_1) - \bar{\tau}_1(\mathrm{d}\tau_2 \mp \rho c_e \,\mathrm{d}v_2) = 0 \quad \text{along} \quad c = \pm c_e, \tag{61a}$$

$$\bar{\tau}_1(\mathrm{d}\tau_1 \mp \rho c_p \,\mathrm{d}v_1) + \bar{\tau}_2(\mathrm{d}\tau_2 \mp \rho c_p \,\mathrm{d}v_2) = 0 \quad \text{along} \quad c = \pm c_p, \tag{61b}$$

which are similar to equations (13). The characteristic conditions along $c = 0$ simply reduce to equations (55), (56a) and (56b).

The simple wave solution can be obtained in a similar manner. Omitting the derivations, we will present the result by an example. Suppose that the material is initially prestressed to the stress state indicated by point i in Fig. 8(b) which is on a new yield circle also given and, at $t = 0$, the stress state at $x = 0$ is changed to point b as shown in Fig. 8(b). The simple wave solution will consist of three regions of constant state with two shock waves as the dividing lines (Fig. 8a). The stress state in the region between OA and OB is the point a in Fig. 8(b) which is located by the intersection of the yield circle and the line connecting point b to the center of the yield circle. The deformation process from i to a is elastic while from a to b is plastic. The yield circles for the regions i and b of Fig. 8(a) are different as shown in Fig. 8(b).

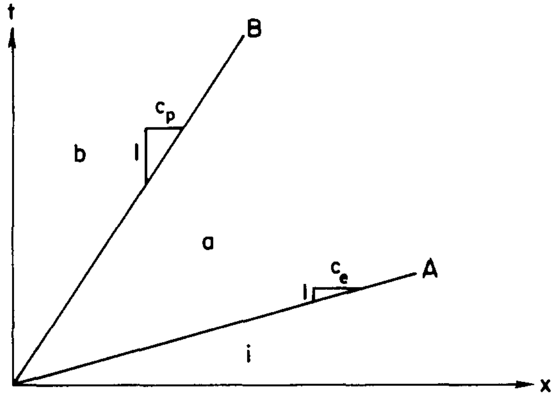

FIG. 8a

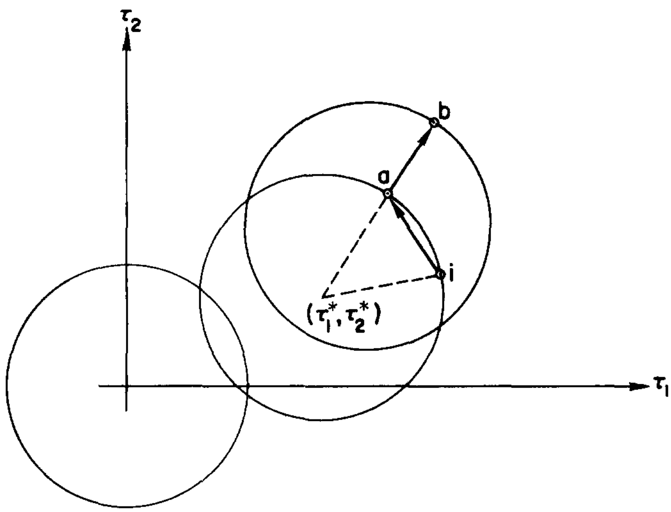

FIG. 8b

One can proceed, as in the isotropic work-hardening case, to obtain shock wave interactions for the kinematical work-hardening materials. Since the procedure is straight forward, this is omitted here.

8. DISCUSSION

Although a series of step loadings and unloadings may not occur in practical applications, the problem considered here offers an exact, closed form solution which is not possible for general loadings. The analysis can be applied to a plate of finite thickness with two or more layers of different materials. It also offers some qualitative results. As is shown in an example, any change in the stress state at one surface of the plate, regardless of whether the change is from a stress state at a lower yield surface to a higher yield surface or vice versa, the disturbance is always (with one exception) propagated at the elastic wave speed. The exceptional case is the one in which the two shear wave can be reduced to a single shear wave. This phenomenon, which is usually attributed to the rate sensitivity of the material, is due to the combined stresses even though the material is not rate dependent. Moreover, this phenomenon exists regardless of whether the material is isotropic work-hardening or kinematical work-hardening.

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REFERENCES

[1] KH. A. RAKHMATULIN, On the propagation of elastic–plastic waves owing to combined loadings. *PMM* **22**, 1079 (1958).
[2] N. CRISTESCU, On the propagation of elasto-plastic waves for combined stresses. *PMM* **23**, 1605 (1959).
[3] H. H. BLEICH and I. NELSON, Plane waves in an elastic–plastic half-space due to combined surface pressure and shear. *J. appl. Mech.* **33**, 149 (1966).
[4] R. J. CLIFTON, An analysis of combined longitudinal and torsional plastic waves in a thin-walled tube. *Fifth U.S. Natnl Congr. Appl. Mech.* June 1966, 465–480.
[5] JEFFREY T. FONG, Elastic–plastic wave in a half-space of a linearly work-hardening material for coupled shear loadings, *Technical Report No.* 161, *Division of Engineering Mechanics, Stanford University* (May 1966).
[6] D. L. VITIELLO and R. J. CLIFTON, The numerical solution of a problem in the propagation of plastic waves of combined stresses. *Brown University Technical Report No.* 4 *to U.S. Army Research Office* (July 1967).
[7] R. HILL, *The Mathematical Theory of Plasticity*, Clarendon Press, Oxford (1950).
[8] R. COURANT and D. HILBERT, *Methods of Mathematical Physics*, Vol. II. Interscience (1962).
[9] A. JEFFREY and T. TANIUTI, *Non-linear Wave Propagation with Applications to Physics and Magneto-hydrodynamics*. Academic Press (1964).
[10] W. PRAGER, The theory of plasticity—a survey of recent achievements, *Proc. Inst. mech. Engrs* No. 169, 41 (1955).

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Абстракт—Исследуется распространение плоских, упруго-пластических волн, принадлежащих к комбинированным двум нагрузкам сдвига. Подразумиваются, что материалы являются упругие, изотропные с упрочнением. Для общего закона упрочнения можно построить аналитические решение только для случая простых волн и двух волн сдвига, которые сокращаются до одинарной волны сдвига. Для упругого материала, с линейным упрочнением можно получить решения также для двух волн сдвига, принадлежащих к ряду ступенчатых нагрузок и разгрузок. В решении заключаются ударные волны, распространяющиеся с постоянными скоростьями и постоянном напряженном состоянием между ударами. Исследуется подробно взаимодействие двух или более ударов, встречающихся в точке, а также отражение ударной волны от жесткой поверхности или от поверхности раздела между двумя разными средами. Используются результаты для случая двух волн сдвига в пластинке конечной тощины, изготовленной из двух или более слоев разных метериалов. Показано кратко, что подобные результаты получаются материалов с упругим поведением кинематического упрочнения.